Optimization of GNSS Deformation Monitoring Networks by Considering Baseline Correlations

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SUMMARY

In the study of deformations of man-made constructions or in geodynamics one usually needs to carefully monitor fixed objects attached to the deformable body. The purpose is to use precise observations to build up an accurate, reliable and possibly low-cost network around the objects to study their motion in short- or long-time intervals and to estimate the possible displacements or deformations among those objects. Frequently, such studies are performed to prevent unwanted disasters (e.g. due to earthquakes and landslides as well as the progressive or abrupt destruction of large-scale structures). This study is concerned with designing an optimal GNSS network to monitor possible deformations of a geodetic network.

By considering GNSS observations one can perform the optimization according to some pre-defined criteria and come up with the best location of receivers and optimum number of baselines. In practice, it is quite common to neglect the effect of correlations between baselines, and instead use single-baseline adjusted data in the optimisation procedure. However, in each session of observation usually more than two receivers are simultaneously taking data from a number of common GNSS satellites. This procedure inevitably leads to between-baseline correlations. Our study designs an optimal observation plan for a GNSS monitoring network with the aim of determining possible displacements and deformations. The developed methodology will be tested on a simulated network with five points, where three receivers simultaneously take data from four satellites.

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1. INTRODUCTION

The efficiency and many advantages of a Global Navigation Satellite System (GNSS) in comparison to other conventional surveying techniques made it very popular over the last few years. Reaching to a higher positioning accuracy is a challenge to almost all technologies including GNSS. Among different types of observations in GNSS, the phase double-difference observations yield higher accuracies than undifferenced observations for short and medium length baselines. By using the undifferenced data, the simultaneous observations are independent of each other and thus, there is no correlation in the measurements. However, a number of errors that can be eliminated or reduced by using differenced approaches still remain in undifferenced observations. For example, satellite clock errors, receiver clock errors and phase ambiguities are parameters that can successfully be removed by single, double and triple phase differences.

The double-difference phase observations are subjected to two types of correlations: the mathematical and physical correlations. The former is created due to differencing the phase observations and the latter results from environmental effects such as for instance atmospheric effects. The physical correlations, which are usually neglected in computations, can be of spatial and/or temporal nature. The impact of temporal physical correlations on the accuracy of relative GPS positioning was studied by El-Rabbany and Kleusberg (2003). Although they elucidated the insignificant effect of physical correlations on estimation of coordinates and ambiguities, they found that neglecting the correlation would create an overly opimistic Variance-Covariance (VC) matrix for the network. It has also been shown that as the time of observation increases, the physical correlation decreases the accuracy.

In contrary to the physical correlation that needs complicated empirical models to be presented, the mathematical correlation can be easily formulated by forming the double-difference equations. To deal with this type of correlation, one should either avoid double-difference observations (use undifferenced observations instead) or compute a VC matrix addressing the correlation effects. In order to have a more realistic VC matrix, it is required to take the correlation into acount. In this way, better position estimation and more realistic estimates of the errors can be achieved (Santos, et al., 1997).

Beutler, et al. (1986) introduced an efficient algorithm that significantly simplified the computation of the weight matrix of the double-difference observations by considering the mathematical correlations. The versatility of their approach was even enhanced by handling the missing data from satellites for any baseline. A year after, Beutler, et al. (1987) took the advantage of their own developed algorithm to investigate the effect of mathematical correlations between simultaneous GPS double-difference phase observations. They coped with correlations by different assumptions; neglecting correlations, considering correlations within single baselines and consideing between-baseline correlations. The results showed that unless the utmost precision was required for a network, all those three assumed models yielded rather similar outcomes.

Optimization of GNSS networks plays an important role in designing an observation plan, where all required precision and reliability and sensitivity criteria are fulfilled. It is quite common in such studies to optimize a network without considering correlations between baselines (see e.g. Kuang 1996, Shestakov, et al. 2005, Mehrabi & Voosoghi 2014). Despite of the minor effect of correlations in the estimation of station coordinates, it can be of importance for deformation monitoring purposes, where a very high level of accuracy is demanded.

In this paper the effect of mathematical correlations between baselines is considered in optimization of a simulated GNSS network. Among several design stages pioneered by Grafarend (1974) that lead to optimal networks, we use the First- and Second-Order Designs (FOD, SOD) to develope a GNSS deformation monitoring network by considering the precision and reliability criteria. In contrast to the typical SOD problems, where the weight of each baseline is subjected to be optimized, the variance factor of each session of observations, which consists of more than one baseline, will be optimized here.

2. METHODOLOGY

In order to investigate the effect of correlations between GNSS baselines in designing and optimizing a GNSS network, a simulated network with five points is considered. We assume that there are three receivers, which can simultaneously take data from four satellites. In this model point *a* is a known point, so that the receiver at point *a* is a reference receiver. The other two receivers can be moved among the unknown network points (denoted *b*, *c*, *d*, *e*) to collect data in several observation sessions. Satellite *j* is also taken as a reference satellite to form a set of double-difference phase observation equations. The other satellites are denoted *l*, *m* and *n*. As we are working with only phase observations, in each session at least two epochs of observations are needed to overcome the number of unknown parameters in the adjustment procedure.

2.1 Network Adjustment by Double-Difference Observations

A phase double-difference equation for two receivers *a* and *b* that takes data from satellites *j* and *k* in epoch *t* can be formulated as (Hofmann-Wellenhof, et al., 2008, p. 255):

$$\lambda\Phi_{ab}^{jk}\left(t\right)=\rho_{ab}^{jk}\left(t\right)+\lambda N_{ab}^{jk} \tag{1}$$

where λ is the carrier phase wavelength and N represents the double-difference phase ambiguity. Φ_{ab}^{jk} is the double-difference observation, consisting of four phase observations $\left(\Phi_r^s\right)$ as:

$$\Phi_{ab}^{jk}(t)=\Phi_{b}^{k}(t)-\Phi_{b}^{j}(t)-\Phi_{a}^{k}(t)+\Phi_{a}^{k}(t) \quad (2)$$

and $\rho_{ab}^{jk}(t)$ is the double-difference geodetic range between receivers and satellites:

$$\rho_{ab}^{jk}(t)=\rho_{b}^{k}(t)-\rho_{b}^{j}(t)-\rho_{a}^{k}(t)+\rho_{a}^{j}(t). \quad (3)$$

By linearizing the terms of Eq. (3) according to unknown coordinates of point *b*, and inserting the linearized form of this equation into Eq. (1), the observation equation at epoch *t* can be formulated as:

$$\underbrace{\lambda\Phi_{ab}^{jk}-\rho_{b0}^{k}+\rho_{b0}^{j}+\rho_{a}^{k}-\rho_{a}^{j}}_{l_{ab}^{jk}}=\mathrm{A}_{\mathrm{x}_b}^{jk}\,\Delta\mathrm{x}_b+\mathrm{A}_{\mathrm{y}_b}^{jk}\,\Delta\mathrm{y}_b+\mathrm{A}_{\mathrm{z}_b}^{jk}\,\Delta\mathrm{z}_b+\lambda N_{ab}^{jk} \quad (4)$$

where ρ_{r0}^{s} is the approximate range values between the receiver *r* and satellite *s*. All the terms in the left side of the equation are known, while the right side of the equation consists of the unknown receiver coordinate improvements ($\Delta\mathrm{x}_r, \Delta\mathrm{y}_r$ and $\Delta\mathrm{z}_r$) . The coefficients $\mathrm{A}_{\mathrm{x}_b}^{jk}$, $\mathrm{A}_{\mathrm{y}_b}^{jk}$ and $\mathrm{A}_{\mathrm{z}_b}^{jk}$ are the evaluated partial derivatives of the range functions at the approximate coordinates of receiver *b*. The numerical values of these coefficients change between observation epochs, as the coordinates of available satellites change with time.

Equation (4) can be established for each pair of receivers and satellites. For instance, by considering the above assumption of three receivers and four satellites in one session, one obtains six double-difference observation equations in each epoch (t_1). In addition to six unknown coordinates for the whole system of equations, there is one unknown ambiguity for each observation that increases the number of unknowns to twelve. If the goal is to perform an adjustment to estimate the unknown coordinates and the float ambiguities, one can either include six more observations from a second epoch (t_2) and/or from code observations for the same epoch (t_1). The general observation equations and the details of the design matrix can be expressed as follows:

$$\mathbf{l}_i-\boldsymbol{\varepsilon}_i=\mathbf{A}_i\,\mathbf{x}_i, \quad i=1,2,...,n \quad (5)$$

with (details only for session i = 1)

$$\mathbf{l}_1=\left[l_{ab}^{jk}(t_1)\quad l_{ab}^{jl}(t_1)\quad\cdots\quad l_{ac}^{jm}(t_1)\quad l_{ab}^{jk}(t_2)\quad l_{ab}^{jl}(t_2)\quad\cdots\quad l_{ac}^{jm}(t_2)\right]_{12\times1}^{\mathrm{T}} \quad (6)$$

$$\mathbf{A}_1=\begin{bmatrix} \mathrm{A}_{\mathrm{x}_b}^{jk}(t_1) & \mathrm{A}_{\mathrm{y}_b}^{jk}(t_1) & \mathrm{A}_{\mathrm{z}_b}^{jk}(t_1) & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ \mathrm{A}_{\mathrm{x}_b}^{jl}(t_1) & \mathrm{A}_{\mathrm{y}_b}^{jl}(t_1) & \mathrm{A}_{\mathrm{z}_b}^{jl}(t_1) & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ \mathrm{A}_{\mathrm{x}_b}^{jm}(t_1) & \mathrm{A}_{\mathrm{y}_b}^{jm}(t_1) & \mathrm{A}_{\mathrm{z}_b}^{jm}(t_1) & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{A}_{\mathrm{x}_c}^{jk}(t_1) & \mathrm{A}_{\mathrm{y}_c}^{jk}(t_1) & \mathrm{A}_{\mathrm{z}_c}^{jk}(t_1) & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \mathrm{A}_{\mathrm{x}_c}^{jl}(t_1) & \mathrm{A}_{\mathrm{y}_c}^{jl}(t_1) & \mathrm{A}_{\mathrm{z}_c}^{jl}(t_1) & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mathrm{A}_{\mathrm{x}_c}^{jm}(t_1) & \mathrm{A}_{\mathrm{y}_c}^{jm}(t_1) & \mathrm{A}_{\mathrm{z}_c}^{jm}(t_1) & 0 & 0 & 0 & 0 & 0 & \lambda \\ & & & & & (t_2) & & & & & & \end{bmatrix}_{12\times12} \tag{7}$$

$$\mathbf{x}_1=\left[\Delta\mathrm{x}_b\ \Delta\mathrm{y}_b\ \Delta\mathrm{z}_b\ \Delta\mathrm{x}_c\ \Delta\mathrm{y}_c\ \Delta\mathrm{z}_c\ N_{ab}^{jk}\ N_{ab}^{jl}\ N_{ab}^{jm}\ N_{ac}^{jk}\ N_{ac}^{jl}\ N_{ac}^{jm}\right]_{12\times1}^{\mathrm{T}} \tag{8}$$

and $\boldsymbol{\varepsilon}_1$ is the vector of residuals. The subscript i represents the number of an observation session in the network, and n is the total number of all possible combinations of sessions in the network that can be made by three receivers.

2.2 Correlated Weight Matrix

A mathematical correlation in a GNSS network is created due to simultaneous double-difference phase observations. The coefficients of Eq. (2) for a number of double-difference observations (now denoted $\nabla\Delta\Phi$) form a matrix **C**, which shows the relation of phase observables (Φ) by 1, -1 and 0. Introducing **C**, Eq. (2) for three receivers and four satellites reads:

$$\nabla\Delta\boldsymbol{\Phi}_{6\times1}=\mathbf{C}_{6\times12}\ \boldsymbol{\Phi}_{12\times1}. \tag{9}$$

By applying the law of error propagation to Eq. (9), the Variance-Covariance (VC) matrix of double-difference observations follows by:

$$\sum\nolimits_{\nabla\Delta}=\mathbf{C}\sum\nolimits_{\boldsymbol{\Phi}}\mathbf{C}^{\mathrm{T}} \tag{10}$$

with

$$\sum\nolimits_{\boldsymbol{\Phi}}=\sigma^2\mathbf{I} \tag{11}$$

being the VC matrix of the phase observables. σ^2 is the variance factor and **I** is a unit matrix. Inserting Eq. (11) into (10) yields:

$$\sum\nolimits_{\nabla\Delta}=\sigma^2\,\mathbf{C}\,\mathbf{C}^{\mathrm{T}}=\mu\,\mathbf{C}\,\mathbf{C}^{\mathrm{T}}. \tag{12}$$

The correlated weight matrix **P** of observations can be numerically obtained for any session i as:

$$\mathbf{P}_i = \sum\nolimits_{\nabla\Delta}^{-1} = \left(\mu_i\,\mathbf{C}\mathbf{C}^{\mathrm{T}}\right)^{-1} = \frac{1}{\mu_i}\begin{bmatrix} 4 & 2 & 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 1 & 2 & 1 \\ 2 & 2 & 4 & 1 & 1 & 2 \\ 2 & 1 & 1 & 4 & 2 & 2 \\ 1 & 2 & 1 & 2 & 4 & 2 \\ 1 & 1 & 2 & 2 & 2 & 4 \end{bmatrix}^{-1}. \tag{13}$$

As can be seen in Eq. (13) **P** is a fully populated matrix that indicates the existing correlations between two baselines. Ignoring this phenomenon yields a diagonal matrix with zero off-diagonal elements for **P**.

The number of possible session combinations of *r* receivers (GNSS sessions) in a network with *m* points is $n = \binom{m}{r} = \frac{m!}{r!\left(m-r\right)!}$. However, if we assume some fixed points in the network, the number of sessions will be decreased. To include the observation equations of all sessions in the network adjustment, block diagonal matrices are formed for matrices **A** and **P** to compute the VC matrix $\mathbf{C}_{\Delta\mathrm{x}}$ of the network:

$$\mathbf{C}_{\Delta\mathrm{x}} = \left(\mathbf{A}^{\mathrm{T}}\,\mathbf{P}\,\mathbf{A}\right)^{-1} = \mathrm{blkdiag}\left[\left(\mathbf{A}_1^{\mathrm{T}}\mathbf{P}_1\mathbf{A}_1\right)^{-1}, \left(\mathbf{A}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{A}_2\right)^{-1}, \ldots, \left(\mathbf{A}_n^{\mathrm{T}}\mathbf{P}_n\mathbf{A}_n\right)^{-1}\right] \tag{14}$$

because

$$\mathbf{A} = \mathrm{blkdiag}\left(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\right) \tag{15}$$

and

$$\mathbf{P} = \mathrm{blkdiag}\left(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n\right). \tag{16}$$

Hence, the VC matrix consists of *n* blocks, where each block contains the variance and covariances of unknown points and ambiguities. Since the goal is to use the obtained VC matrix as an input correlated weight matrix for the baseline adjustment procedure, one can extract a block of the matrix $\mathbf{C}_{\Delta\mathrm{x}}$, which is related to the unknown coordinates, and consequently ignore the ambiguities. It should also be mentioned that since the unknown coordinates are not estimated in the context of network design and optimization, it is needless to resolve the ambiguities.

2.3 Baseline Adjustment

The baseline vector $\mathbf{\Delta X}_{pq} = \left[\Delta\mathrm{x}_{pq} \quad \Delta\mathrm{y}_{pq} \quad \Delta\mathrm{z}_{pq}\right]^{\mathrm{T}}$ between the unknown points $\mathbf{X}_p = \left[\mathrm{x}_p \quad \mathrm{y}_p \quad \mathrm{z}_p\right]^{\mathrm{T}}$ and $\mathbf{X}_q = \left[\mathrm{x}_q \quad \mathrm{y}_q \quad \mathrm{z}_q\right]^{\mathrm{T}}$ are considered as observables in the least squares baseline adjustment procedure, and their linear relation can be expressed as:

$$\Delta\mathbf{X}_{pq} = \mathbf{X}_q - \mathbf{X}_p \tag{17}$$

where p and q are the network points.

The known baselines here are the unknowns of the previous section, where they are supposed to be determined. Therefore, the VC of the coordinates from the previous section can be introduced as weights of observables in this adjustment. Assuming that each session of observations in a GNSS network has three points and point a is a fixed one, the observation equations (Eq. (5)) for two independent baselines of first session can be written as:

$$\begin{bmatrix} \Delta x_{ab} \\ \Delta y_{ab} \\ \Delta z_{ab} \\ \Delta x_{ac} \\ \Delta y_{ac} \\ \Delta z_{ac} \end{bmatrix}_{6\times1} - \boldsymbol{\varepsilon}'_1 = \begin{bmatrix} x_b - x_a \\ y_b - y_a \\ z_b - z_a \\ x_c - x_a \\ y_c - y_a \\ z_c - z_a \end{bmatrix}_{6\times1} - \boldsymbol{\varepsilon}'_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times9} \begin{bmatrix} x_a \\ y_a \\ z_a \\ x_b \\ y_b \\ z_b \\ x_c \\ y_c \\ z_c \end{bmatrix}_{9\times1} = \mathbf{D}_1\mathbf{x}'_1 \tag{18}$$

where $\mathbf{D}_1$ is the design matrix that describes the relation between observations and unknowns. The residual and unknown vectors are shown by $\boldsymbol{\varepsilon}'$ and $\mathbf{x}'$, respectively. It should be noted in Eq. (18) that although the unknown vector contains the coordinates of point a, its coordinates are already assumed known, and it is just included to be able to form the design matrix. This point can be eliminated from the adjustment procedure by using the minimum constraints, where point a will be opted as fixed. Hence, the datum deficiency can be resolved by defining the minimum constraint (**B**) and inner constraint (**E**) matrices as (Kuang, 1996, pp. 102-108):

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}_{3\times9} \tag{19}$$

and

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}_{3\times9}. \tag{20}$$

The weight matrix $\mathbf{Q}_i$ for the observation session i can be defined as the inverted VC matrix (see Eq. 14):

$$\mathbf{Q}_i = \mathbf{A}_i^{\mathrm{T}} \mathbf{P}_i \mathbf{A}_i = \frac{1}{\mu_i} \mathbf{A}_i^{\mathrm{T}} \left(\mathbf{C}\mathbf{C}^{\mathrm{T}} \right)^{-1} \mathbf{A}_i, \quad i = 1, 2, ..., n \tag{21}$$

and the VC matrix for all sessions in the network can be written as (Kuang, 1996, p. 221):

$$\mathbf{C}_{\mathrm{x}} = \left(\mathbf{D}^{\mathrm{T}} \mathbf{Q} \mathbf{D} + \mathbf{B}\mathbf{B}^{\mathrm{T}} \right)^{-1} - \mathbf{E} \left(\mathbf{E}^{\mathrm{T}} \mathbf{B}\mathbf{B}^{\mathrm{T}} \mathbf{E} \right)^{-1} \mathbf{E}^{\mathrm{T}} \tag{22}$$

where **D**, **Q**, **B** and **E** are the block diagonal matrices. Here the submatrices are respectively, design, weight, minimum and inner constraints matrices for each session of observations.

2.4 Optimization Procedure

Optimizing a geodetic network with respect to some criteria enables us to eliminate a number of unnecessary observations that has no significant effect in improving the demanded precision, reliability and sensitivity of the network. Among different optimization and design stages, usually the FOD step is ignored for GNSS networks due to insignificant role of network configuration in improving its precision. It has been proved by the authors of this paper that moving the points in a GNSS network cannot lead to a substantial change in the VC of adjusted network. However, the SOD problem is relevant and should be solved for such a network to realize which observations are needed to fulfill the defined criteria.

Nowadays, it is very rare to use only two receivers per session of observations. Therefore, working with more than two receivers at a time obviously leads to a correlation between baselines. Traditionally, most of the GNSS software packages perform the baseline adjustment without considering the correlation. With this in mind, the weight matrix cannot be assumed as a diagonal matrix any more, and the effect of mathematical correlation should be formulated (see Section 2.2). In our previous experiences, the output of GNSS network optimization was the unnecessary baselines, which could be removed from an observation plan. Thus, the weight of each baseline was of interest to be optimized and eventually, the zero (or close to) weights indicate the elimination of that baseline. Yet not only a set of baselines are optimized by considering between-baseline correlation, but sets of sessions will be optimized. Here, the magnitude of the optimized variance factor for each session satisfies the pre-defined criteria.

2.5 Bi-Objective Optimization Model (BOOM)

Considering the two important criteria of precision and reliability in optimizing a network, one can design a precise network that is capable to detect possible gross errors. The presented VC matrix in Eq. (22) can be linearized by Taylor series to simplify the optimization procedure as:

$$\mathbf{C}_{\mathrm{x}} = \mathbf{C}_{\mathrm{x}_0} + \sum_{i=1}^{n} \frac{\partial \mathbf{C}_{\mathrm{x}}}{\partial \mu_i} \Delta\mu_i \tag{23}$$

with

$$\frac{\partial \mathbf{C}_{\mathrm{x}}}{\partial \mu_i} = -\left(\mathbf{D}^{\mathrm{T}}\mathbf{Q}\,\mathbf{D} + \mathbf{B}\,\mathbf{B}^{\mathrm{T}}\right)^{-1}\left(\mathbf{D}^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial \mu_i}\mathbf{D}\right)\left(\mathbf{D}^{\mathrm{T}}\mathbf{Q}\,\mathbf{D} + \mathbf{B}\,\mathbf{B}^{\mathrm{T}}\right)^{-1} \tag{24}$$

and $\mathbf{C}_{\mathrm{x}_0}$ being an approximate VC matrix. In this equation, the partial derivatives of the weight matrix $\mathbf{Q}$ with respect to μ_i are block diagonal matrices with zero elements except for the corresponding block diagonal matrix to session *i*. In other words,

$$\frac{\partial \mathbf{Q}}{\partial \mu_i} = -\mu_i^{-2}\mathbf{A}_i^{\mathrm{T}}\left(\mathbf{C}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{A}_i \tag{25}$$

for the *i-th* session on the *i-th* block diagonal matrix, otherwise it follows that the derivative vanishes.

The goal is to fit the VC matrix of the network to a pre-defined criterion. The criterion matrix can be constructed based on the user's requirements. For deformation monitoring purposes, the criterion can be assigned to the optimization procedure such that it increases the sensitivity of the network in detecting deformations (see Alizadeh-Khameneh et al. 2015).

A high reliability of a network guarantees its power in detecting gross errors and minimizing the effect of such errors in the network. Thus, not only precision criterion is needed for a network, but also a reliability criterion is inevitable to strengthen it against blunders and other gross errors. This criterion is associated with the observation redundancy matrix **R**, which can be written as (Kuang, 1996, p. 122):

$$\mathbf{R} = \mathbf{I} - \mathbf{D}\left(\mathbf{D}^{\mathrm{T}}\mathbf{Q}\mathbf{D} + \mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{Q} \tag{26}$$

Expanding Eq. (26) by a Taylor series around its approximate value $\mathbf{R}^0$ yields:

$$\mathbf{R} = \mathbf{R}^0 + \sum_{i=1}^{n}\frac{\partial \mathbf{R}}{\partial \mu_i}\Delta\mu_i \tag{27}$$

with

$$\frac{\partial \mathbf{R}}{\partial \mu_i} = \mathbf{D}\left(\mathbf{D}^{\mathrm{T}}\mathbf{Q}\mathbf{D} + \mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1}\mathbf{D}^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial \mu_i}\left[\mathbf{D}\left(\mathbf{D}^{\mathrm{T}}\mathbf{Q}\mathbf{D} + \mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1}\mathbf{D}^{\mathrm{T}} - \mathbf{I}\right]. \tag{28}$$

These two criteria are brought together as one objective function in a bi-objective optimization model to fulfill the requirements of the network from both precision and reliability points of view. This object function can be mathematically introduced as (Kuang, 1996, p. 253):

$$\left[\frac{\left\|\mathbf{H}\mathbf{w} - \mathbf{u}\right\|}{\left\|\mathrm{vec}\left(\mathbf{C}_{\mathrm{s}}\right)\right\|} + \frac{\left\|\mathbf{R}_2\mathbf{w} - \left(\mathbf{r}_{\mathrm{o}} - \mathbf{R}_1\right)\right\|}{\left\|\mathbf{r}_{\mathrm{o}}\right\|}\right] \to \min \tag{29}$$

subject to physical constraints to assure obtaining non-negative values for μ_i of each session. The first term in Eq. (29) represents the precision, and the other term expresses the reliability objective functions. Using a single model of either of these models will lead to some contradictions and inconsistencies, while a combined model can overcome all such drawbacks. The **H** matrix in the precision term contains the derivatives of the matrix $\mathbf{C}_x$ with respect to μ_i. **u** is a vector that carries the difference between the VC matrix and the criterion matrix $\mathbf{C}_s$. Finally, the elements of improvement of the variance factor ($\Delta\mu_i$) define the vector **w**. In the second term $\mathbf{R}_2$, $\mathbf{R}_1$ and $\mathbf{r}_o$ are a matrix formed by differentiation of the redundancy matrix with respect to μ , a vector including the diagonal elements of the observation reliability matrix, and a vector that introduces the desired value of the reliability into the optimization process, respectively (for more details, see Alizadeh-Khameneh et al. 2015).

3 NUMERICAL STUDIES

As already mentioned in Section 2, the developed methodology is tested on a simulated network with five points. In order to realize the effect of between-baseline correlations in optimization of a GNSS network, three receivers are assumed to simultaneously take data from four satellites in two epochs with an interval in time of 10 minutes. The coordinates of satellites are illustrated in Table 1.

Table 1. Coordinates of available satellites for the simulated network.

	Epoch 1: 2004 2 2 1 0 0.0			Epoch 2: 2004 2 2 1 10 0.0		
Satellite	X (m)	Y (m)	Z (m)	X (m)	Y (m)	Z (m)
24 (*j*)	23421962.76	-12587454.89	999742.384	23473743.85	-12450871.41	-916900.183
13 (*k*)	7552200.21	22602205.12	11771799.88	7448953.44	23436633.99	10096430.39
8 (*l*)	19782985.99	3245004.08	17378780.21	18613894.53	3987630.62	18451242.94
21 (*m*)	-11894220.66	-10766056.10	21172793.12	-10464044.48	-11551201.71	21487805.34

As a first step in optimization of this network, the FOD problem is aimed to be solved. In this stage the optimum locations for the network points are sought such that higher precision can be obtained for the station coordinates. Usually, the FOD is not performed for GNSS networks due to the insignificant effect of moving observation stations on the precision of estimated coordinate. Moving the points on the ground for several meters or even kilometers versus the satellite constellation do not have any notable influence on the satellite-receiver configuration of a local or regional GNSS network. With this in mind, one can see in Fig. 1 that the error ellipses of the network points are almost the same except for point *a* (which is the fixed point). It needs to be clarified that in this simulated network the points are located within a few kilometers from each other. Therefore, the estimated positions of stations are somehow within the same range. Undoubtedly, dealing with continental-scale networks would require the first order design, which is not the case here.

Contrary to the traditional mode of the SOD in designing a network, where the weight of observations are subjected to be optimized, the variance factors of the observation sessions are the subject for optimization in this study. As there is more than one baseline in each session, the mathematical correlation between baselines can be taken into account in the equations. This leads to more realistic results in performing adjustment or optimization of a network. An important step in solving the SOD problem is to define a proper criterion matrix or value to be reached. In this study, the precision criterion matrix is developed to enable the network to detect 4 mm displacements in all directions at each point. A redundancy number for each observation can vary from 0 to 1, indicating a low to highly reliable observation. With this in mind, the criterion value for the reliability is set to 0.7 for each observation.

Implementing these two criteria in a bi-objective function as described in Section 2.5 resists any possible inconsistencies that can happen if these criteria would be used separately. On the other hand, a BOOM guaranties the desired precision and reliability of a network. Therefore, a BOOM of precision and reliability is used here to optimize a simulated GNSS deformation monitoring network. The configuration of the network and the six observation sessions are illustrated with Fig. 1. It can be seen that among these six sessions, the sessions with a-b-c and a-d-e receivers can be eliminated from observation plan. Although performing those sessions may increase the precision and reliability of the network, they are not required according to our pre-defined criteria.

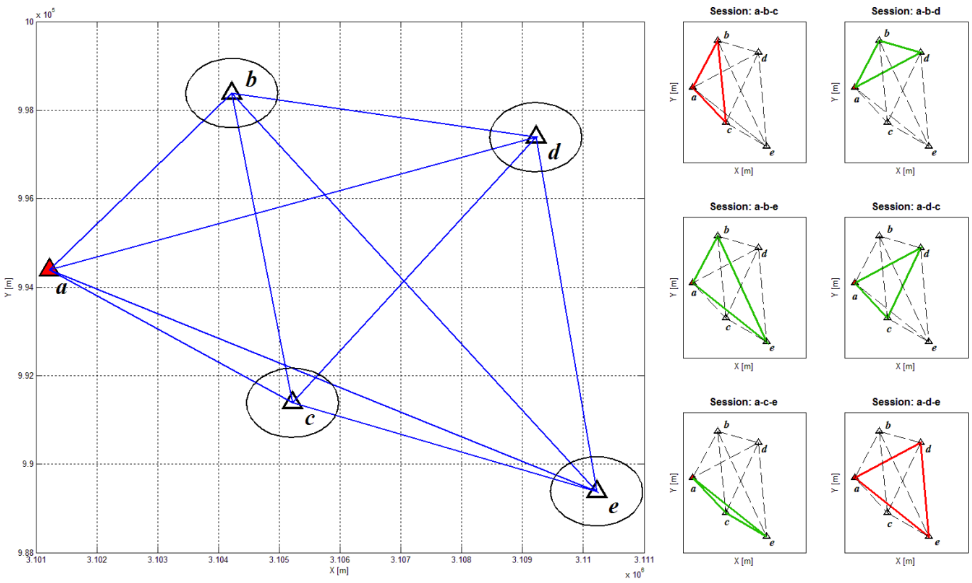

Figure 1. Simulated network with five points, where point “*a*” is assumed to be known. The six subplots on the right side illustrate the possible observation sessions with three receivers in this network (the size of plotted error ellipses are exaggerated for better illustration).

Table 2 shows that the optimization procedure was started by assuming 1 as an initial value for the variance factor of each session. The value after optimization is provided in the fourth column. An extremely large value of the variance factor, which can be seen for the first and last session, leads to zero corresponding weight matrices, and shows the unnecessity of these sessions in the observation plan. By removing those sessions, the number of observed baselines can be reduced from 12 baselines to 8.

Table 2. GNSS sessions and their corresponding optimized variance factors.

Sessions	Observed baselines	Initial variance factor (μ^0)	Optimized value (μ)	Weight matrix
a-b-c	ab, ac	1.00	3.59 e 61	$\mathbf{Q}_{\text{abc}} \approx 0$
a-b-d	ab, ad	1.00	1.87	$\mathbf{Q}_{\text{abd}} = \frac{1}{\mu_{\text{abd}}} \mathbf{A}_{\text{abd}} \left(\mathbf{CC}^{\text{T}} \right)^{-1} \mathbf{A}_{\text{abd}}$
a-b-e	ab, ae	1.00	0.84	$\mathbf{Q}_{\text{abe}} = \frac{1}{\mu_{\text{abe}}} \mathbf{A}_{\text{abe}} \left(\mathbf{CC}^{\text{T}} \right)^{-1} \mathbf{A}_{\text{abe}}$
a-c-d	ac, ad	1.00	1.00	$\mathbf{Q}_{\text{acd}} = \mathbf{A}_{\text{acd}} \left(\mathbf{CC}^{\text{T}} \right)^{-1} \mathbf{A}_{\text{acd}}$
a-c-e	ac, ae	1.00	1.44	$\mathbf{Q}_{\text{ace}} = \frac{1}{\mu_{\text{ace}}} \mathbf{A}_{\text{ace}} \left(\mathbf{CC}^{\text{T}} \right)^{-1} \mathbf{A}_{\text{ace}}$
a-d-e	ad, ae	1.00	1.62 e 77	$\mathbf{Q}_{\text{ade}} \approx 0$

4 CONCLUSION

The objective of this study was to design an optimal GNSS deformation monitoring network by considering the correlations between baselines. For this purpose, a simulated network was developed with five points, and it was assumed that in each observation session, three receivers could simultaneously take data from four satellites. The mathematical correlation was modeled as a weight matrix for each session. Due to the less significant effect of the precision criterion in a small-scale GNSS network, it was decided to add the reliability criterion to the optimization procedure. The variance factor of each session was subjected to be optimized. Performing the BOOM of precision and reliability, two of six sessions received large values for their variance factor, and consequently could be removed from the observation plan without compromising the pre-defined criteria of the ability to detect a 4 mm displacement.

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